Integrability of the n–centre problem at high energies

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We consider the n–centre problem of celestial mechanics in $d = 2$ and 3 dimensions.

In this paper we show that for generic configuration of the centres at high energy levels this system is completely integrable by using C^∞ integrals of the motion however it is not integrable in terms of real analytic integrals.

The Hamiltonian function

$$\hat{H} : T^*\hat{M} \to \mathbb{R} \ , \ \ \hat{H}(\vec{p}, \vec{q}) = \frac{1}{2}\vec{p}^2 + V(\vec{q}),$$

with potential

$$V : \hat{M} \to \mathbb{R} \ \ , \ \ V(\vec{q}) = -\sum_{k=1}^{n} \frac{Z_k}{\|\vec{q} - \vec{s}_k\|},$$

on the cotangent bundle $T^*\hat{M}$ of configuration space

$$\hat{M} := \mathbb{R}^d \setminus \{\vec{s}_1, \ldots, \vec{s}_n\}$$

generates a – in general incomplete – flow. Here $\vec{s}_k \in \mathbb{R}^d$ is location of the k-centre, $\vec{s}_k \neq \vec{s}_l$ for $k \neq l$, and $Z_k \in \mathbb{R} \setminus \{0\}$, $k = 1, \ldots, n$. If $Z_k > 0$ for all k, then this problem describes the motion of a massive particle in the gravitational field of n fixed centres.

In [1] it was, in particular, shown that this system admits a smooth extension (P, ω, H) such that the corresponding flow is complete.

Until recently it was known that

1) for $n = 1$ this system is integrable, with the angular momentum for dimension $d = 3$ being a real analytic constant of motion (for $Z_1 > 0$ this is the Kepler problem);

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2) for $n = 2$ this system is integrated by using the elliptic prolate coordinates (this was done by Euler);

3) for $n \geq 3$ centres and $d = 2$ it is showed in [2] that there is no analytic integral of the motion which is non–constant on an energy shell $H^{-1}(E)$, $E > 0$;

4) for $d = 3$ and a collinear configuration of centres the angular momentum w.r.t. that axis is an additional constant of the motion, independent of the number n of centres;

5) for $d = 3$ it was proved that the topological entropy of the flow restricted to the set of bounded orbits b_E is positive ([1] for sufficiently large energies $E > E_{\text{th}}$, [3] for nonnegative energies $E \geq 0$) and $h_{\text{top}} = 0$ if b_E is empty. Furthermore $h_{\text{top}}(E)$ vanishes for $n = 1$ and 2, and $h_{\text{top}}(E) > 0$ if $n \geq 3$ and all centres being attracting or not more than two $\vec{s}_k$ being on a line (for collinear configurations with $Z_1, \ldots, Z_n < 0$ one has $h_{\text{top}}(E) = 0$ for $E > 0$).

Orbits of the flow fall into three classes: bounded, scattering, and trapped. The subsets formed by these orbits are defined by b, s, and t respectively. The limits of scattering orbits are described by comparison with the Kepler flow generated by the extension of

$$\hat{H}_\infty : T^*(\mathbb{R}^d \setminus \{0\}) \to \mathbb{R}, \quad \hat{H}_\infty(\vec{p}, \vec{q}) := \frac{1}{2}\vec{p}^{\,2} - \frac{Z_\infty}{\|\vec{q}\|}, \quad Z_\infty = \sum_{k=1}^{n} Z_k.$$

It was proved in [1] that the set of trapped orbits is of measure zero and

- for $d = 2$ and attracting centres $(Z_k > 0)$;

- for $d = 3$, arbitrary $Z_k \neq 0$ and noncollinear configurations of centres

there is a threshold energy $E_{\text{th}} \geq 0$ such that

- for $E > E_{\text{th}}$ many estimates are proved and, in particular, the set b_E of bounded orbits is of measure zero;

- therefore above this threshold energy almost every point x in the phase space lies on a scattering orbit and the following smooth functions are defined on he set formed by scattering orbits:

  a) the asymptotic limits of the momentum:

$$\vec{p}^{\,\pm} : s \to \mathbb{R}^d;$$

b) the time delay

$$\tau : s \to \mathbb{R}$$

which is the asymptotic difference between the time spent by the orbit passing through x and its Kepler limit inside a ball of large radius. This function diverges near $b \cup t$.

We use these results and in the sequel assume that $d = 2$ and $Z_k > 0$ or $d = 3$ and the configuration of the centres is noncollinear.

It appears that the asymptotyc limits of the momentum gives rise to integrals of the motion. We have

Theorem 1 *For any $E_1, E_2 > E_{\mathrm{th}}$ with $E_1 \leq E_2$, there exists a constant $C > 0$ such that for any $g > 1$ the functions $f_k^g : H^{-1}([E_1, E_2]) \to \mathbb{R}$ of the form*

$$f_k^g(x) := \begin{cases} p_k^+(x) \exp\left(-e^{\frac{C}{g-1}\sqrt{1+\tau^2(x)}}\right) & , \quad x \in s \\ 0 & , \quad x \notin s, \end{cases}$$

are functionally independent of a full measure subset, integrals of the motion and are of the Gevrey class of index g.

Corollary 1 *On every submanifold $H^{-1}((E_1, E_2))$ with $E_2 > E_1 > E_{\mathrm{th}}$ the n-centre problem is completely integrable.*

As we use the same trick as the one used in [4, 5] it is obvious that a similar result concerning Gevrey integrability of these systems can be obtained. In particular, [5] an example of the integrable geodesic flow with positive topological entropy on a compact real analytic Riemannian manifold was constructed. Remark that in the situation of Theorem 1 the restriction of the n-centre problem onto the set of bounded orbits does not change the positive value of topological entropy [1, 3]. Moreover for large values of E the n-centre is also not analytically integrable as in the example given in [5]:

Theorem 2 *On any level set $H^{-1}(E)$ with $E > E_{\mathrm{th}}$ the n-centre problem does not admit a pair of functionally independent real analytic integrals of motion.*

The obstruction to such an integrability is as follows. Assume that there are such real analytic integrals of motion. Let $P_E = H^{-1}(E)$ and let S be the subset of P_E on which these integrals are functionally dependent. It contains $b_E = P_E \cap b$, i.e. bounded states of this energy. Take a generic point

$x \in S$ and denote by γ the intersection of S with the unstable submanifold of the Poincaré surface. Fix some Riemannian metric on P_E. By using results of [1] it is proved that for $E > E_{\text{th}}$ there have to exist a vector v_∞ which is tangent to γ at x and a sequence $\{v_n\}$ of vectors tangent to P_E at x such that

$$\exp(x, v) \in \gamma, \quad \lim_{n \to \infty} v_n = 0$$

and

$$\frac{\pi}{2} \geq \angle(v_\infty, v_n) \geq O(r_n^{1+\alpha}), \quad r_n = |v_n|,$$

for some constant $\alpha \in (0, 1)$. However the analytic integrability on the level P_E implies that for a generic point $x \in S$ the set γ has to be a one-dimensional manifold. By the Taylor decomposition, this implies that the angles has to converge faster than $r_n^{1+\alpha}$:

$$\angle(v_\infty, v_n) \sim O(r^2), \quad r_n = |v_n|.$$

Thus we arrive at a contradiction which implies Theorem 2.

Proofs of these theorems will be published elsewhere.

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